Solutions to Nine’s Puzzle Challenge II – Allowing other roots beyond √

(rev. 09/21/2021)

There are an unbounded number of solutions to Challenge II. The crucial insight is using roots to generate either the values 2, 4, and 16 (Solution I) or 2, 4, and 8 (Solution II) or 3, and 9 (Solution III).

However, the solution for the integer 8 does not fall into one of the 3 solution cases presented below.

$8 = 2^3$

$$\text{Note: } \sqrt[3]{8} = 2 \text{ and } \frac{8}{\sqrt[3]{8}} = \sqrt{\sqrt[3]{8} \times 8} = 4$$

$1 = \frac{8}{8}$ $\quad 2 = \sqrt[3]{8} = \frac{8+8}{8}$ $\quad 3 = \sqrt[3]{8} + \frac{8}{8}$ $\quad 4 = \frac{8}{\sqrt[3]{8}} = \sqrt{\sqrt[3]{8} \times 8}$ $\quad 5 = 8 - \left(\sqrt[3]{8} + \frac{8}{8}\right)$

$6 = 8 - \sqrt[3]{8}$ $\quad 7 = 8 - \frac{8}{8}$ $\quad 8 = 8$ $\quad 9 = 8 + \frac{8}{8}$ $\quad 10 = 8 + \sqrt[3]{8}$ $\quad 11 = 8 + \sqrt{\sqrt[3]{8} \times 8} - \frac{8}{8}$

$12 = 8 + \frac{8}{\sqrt[3]{8}}$ $\quad 13 = 8 \times \sqrt[3]{8} - \sqrt{8 + \frac{8}{8}}$ $\quad 14 = 8 \times \sqrt[3]{8} - \frac{8+8}{8}$ $\quad 15 = 8 \times \sqrt[8]{8} - \frac{8}{8}$ $\quad 16 = 8 \times \sqrt[3]{8}$

$17 = 8 \times \sqrt[3]{8} + \frac{8}{8}$ $\quad 18 = \left(8 + \frac{8}{8}\right) \times \sqrt[3]{8}$ $\quad 19 = ?$

9’s Clock Solution I - Case $k = 16^n$: Using only the integer k and the algebraic operations of addition, subtractions, multiplication, division, exponentiation, and integer roots (e.g., square root, cube root, etc.) *no more than one time*, if $k = 16^n = 4^{2n} = 2^{4n}$ for $n \geq 1$ then the clock integers 1 thru 12 can be generated.

Proof: The equivalences $\sqrt[4n]{16^n} = \sqrt[4n]{2^{4n}} = 2$, $\sqrt[2n]{16^n} = \sqrt[2n]{4^{2n}} = 4$, and $\sqrt[n]{16^n} = 16$ will be used below noting that $\sqrt[4n]{16^n} \times \sqrt[2n]{16^n} = 2 \times 4 = 8$

$1 = \frac{16^n}{16^n}$ $\quad 2 = \frac{16^n + 16^n}{16^n} = \sqrt[4n]{16^n}$ $\quad 3 = \sqrt[4n]{16^n} + \frac{16^n}{16^n} = \sqrt[2n]{16^n} - \frac{16^n}{16^n}$ $\quad 4 = \sqrt[2n]{16^n}$ $\quad 5 = \sqrt[2n]{16^n} + \frac{16^n}{16^n}$

$6 = \sqrt[2n]{16^n} + \sqrt[4n]{16^n}$ $\quad 7 = \sqrt[4n]{16^n} \times \sqrt[2n]{16^n} - \frac{16^n}{16^n}$ $\quad 8 = \sqrt[4n]{16^n} \times \sqrt[2n]{16^n}$ $\quad 9 = \sqrt[4n]{16^n} \times \sqrt[2n]{16^n} + \frac{16^n}{16^n}$

$10 = \sqrt[4n]{16^n} \times \left(\sqrt[2n]{16^n} + \frac{16^n}{16^n}\right)$ $\quad 11 = \sqrt[n]{16^n} - \left(\sqrt[2n]{16^n} + \frac{16^n}{16^n}\right)$ $\quad 12 = \sqrt[n]{16^n} - \sqrt[2n]{16^n}$

$13 = \sqrt[n]{16^n} - \left(\sqrt[4n]{16^n} + \frac{16^n}{16^n}\right)$ $\quad 14 = \sqrt[n]{16^n} - \sqrt[4n]{16^n}$ $\quad 15 = \sqrt[n]{16^n} - \frac{16^n}{16^n}$ $\quad 16 = \sqrt[n]{16^n}$

$17 = \sqrt[n]{16^n} + \dfrac{16^n}{16^n}$ $18 = \sqrt[n]{16^n} + \sqrt[4n]{16^n}$ $19 = \sqrt[n]{16^n} + \sqrt[2n]{16^n} - \dfrac{16^n}{16^n}$

$20 = \sqrt[n]{16^n} + \sqrt[2n]{16^n}$ $21 = ?$

Example: $16 = 4^2 = 2^4$

Note: $\sqrt{16} = 4$ $\sqrt[4]{16} = 2$

$1 = \dfrac{16}{16}$ $2 = \dfrac{16+16}{16}$ $3 = \sqrt{16} - \dfrac{16}{16}$ $4 = \sqrt{16}$ $5 = \sqrt{16} + \dfrac{16}{16}$ $6 = \sqrt{16} + \sqrt[4]{16}$

$7 = \sqrt[4]{16} \times \sqrt{16} - \dfrac{16}{16}$ $8 = \sqrt[4]{16} \times \sqrt{16}$ $9 = \sqrt[4]{16} \times \sqrt{16} + \dfrac{16}{16}$ $10 = \sqrt[4]{16} \times \left(\sqrt{16} + \dfrac{16}{16}\right)$

$11 = 16 - \left(\sqrt{16} + \dfrac{16}{16}\right)$ $12 = 16 - \sqrt{16}$ $13 = 16 - \sqrt{16} + \dfrac{16}{16}$ $14 = 16 - \sqrt[4]{16}$ $15 = 16 - \dfrac{16}{16}$

$16 = 16$ $17 = 16 + \dfrac{16}{16}$ $18 = 16 + \sqrt[4]{16}$ $19 = 16 + \sqrt{16} - \dfrac{16}{16}$ $20 = 16 + \sqrt{16}$ $21 = ?$

9's Clock SolutionII - Case $k = 64^n$: Using only the integer k and the algebraic operations of addition, subtractions, multiplication, division, exponentiation, and integer roots (e.g., square root, cube root, etc.) *no more than one time*, if $k = 64^n = 8^{2n} = 4^{3n} = 2^{6n}$ for $n \geq 1$ then the clock integers 1 thru 12 can be generated.

Proof: The equivalences $\sqrt[6n]{64^n} = \sqrt[6n]{2^{6n}} = 2$, $\sqrt[3n]{64^n} = \sqrt[3n]{4^{3n}} = 4$, and $\sqrt[2n]{64^n} = \sqrt[2n]{8^{2n}} = 8$ will be used below noting that $16 = \sqrt[6n]{64^n} \times \sqrt[2n]{64^n}$

$1 = \dfrac{64^n}{64^n}$ $2 = \dfrac{64^n + 64^n}{64^n} = \sqrt[6n]{64^n}$ $3 = \sqrt[6n]{64^n} + \dfrac{64^n}{64^n} = \sqrt[3n]{64^n} - \dfrac{64^n}{64^n}$ $4 = \sqrt[3n]{64^n} = 4$ $5 = \sqrt[3n]{64^n} + \dfrac{64^n}{64^n}$

$6 = \sqrt[3n]{64^n} + \sqrt[6n]{64^n}$ $7 = \sqrt[2n]{64^n} - \dfrac{64^n}{64^n}$ $8 = \sqrt[2n]{64^n}$ $9 = \sqrt[2n]{64^n} + \dfrac{64^n}{64^n}$

$10 = \sqrt[2n]{64^n} + \sqrt[6n]{64^n}$ $11 = \sqrt[2n]{64^n} + \sqrt[3n]{64^n} - \dfrac{64^n}{64^n}$ $12 = \sqrt[2n]{64^n} + \sqrt[3n]{64^n}$

$13 = \sqrt[6n]{64^n} \times \sqrt[2n]{64^n} - \sqrt[3n]{64^n} + \dfrac{64^k}{64^k}$ $14 = \sqrt[6n]{64^2} \times \sqrt[2n]{64^n} - \dfrac{64^n + 64^n}{64^n}$ $15 = \sqrt[6n]{64^2} \times \sqrt[2n]{64^n} - \dfrac{64^n}{64^n}$

$16 = \sqrt[6n]{64^2} \times \sqrt[2n]{64^n}$ $17 = \sqrt[6n]{64^2} \times \sqrt[2n]{64^n} + \dfrac{64^n}{64^n}$ $18 = ?$

Example: 64 = 8^2 = 4^3 = 2^6

Note: $\sqrt{64}=8$ $\sqrt[3]{64}=4$ $\sqrt[6]{64}=2$

$1=\frac{64}{64}$ $2=\frac{64+64}{64}=\sqrt[4]{64}$ $3=\sqrt[3]{64}-\frac{64}{64}$ $4=\sqrt[3]{64}$ $5=\sqrt[3]{64}+\frac{64}{64}$ $6=\sqrt[3]{64}+\sqrt[6]{64}$

$7=\sqrt{64}-\frac{64}{64}$ $8=\sqrt{64}$ $9=\sqrt{64}+\frac{64}{64}$ $10=\sqrt{64}+\sqrt[6]{64}$ $11=\sqrt{64}+\sqrt[3]{64}-\frac{64}{64}$

$12=\sqrt{64}+\sqrt[3]{64}$ $13=\sqrt[6]{64}\times\sqrt{64}-\sqrt[3]{64}+\frac{64}{64}$ $14=\sqrt[6]{64}\times\sqrt{64}-\frac{64+64}{64}$

$15=\sqrt[6]{64}\times\sqrt{64}-\frac{64}{64}$ $16=\sqrt[6]{64}\times\sqrt{64}$ $17=\sqrt[6]{64}\times\sqrt{64}+\frac{64}{64}$ $18=?$

9's Clock Solution III – Case $k=9^n$: Using only the integer k and the algebraic operations of addition, subtractions, multiplication, division, exponentiation, and integer roots (e.g., square root, cube root, etc.) *no more than one time*, if $k=9^n=3^{2n}$ for $n\geq 1$ then the clock integers 1 thru 12 can be generated.

Proof: The equivalences $\sqrt[2n]{9^n}=\sqrt[2n]{3^{2n}}=3$ and $\sqrt[n]{9^n}=9$ will be used below

$1=\frac{9^n}{9^n}$ $2=\sqrt[2n]{9^n}-\frac{9^n}{9^n}$ $3=\sqrt[2n]{9^n}$ $4=\sqrt[2n]{9^n}+\frac{9^n}{9^n}$ $5=\sqrt[n]{9^n}-\left(\sqrt[2n]{9^n}+\frac{9^n}{9^n}\right)$ $6=\sqrt[n]{9^n}-\sqrt[2n]{9^n}$

$7=\sqrt[n]{9^n}-\sqrt[2n]{9^n}+\frac{9^n}{9^n}$ $8=\sqrt[n]{9^n}-\frac{9^n}{9^n}$ $9=\sqrt[n]{9^n}$ $10=\sqrt[n]{9^n}+\frac{9^n}{9^n}$ $11=\sqrt[n]{9^n}+\sqrt[2n]{9^n}-\frac{9^n}{9^n}$

$12=\sqrt[n]{9^n}+\sqrt[2n]{9^n}$ $13=?$

Example: 81 = 9^2 = 3^4

Note: $\sqrt[4]{81}=3$ $\sqrt{81}=9$

$1=\frac{81}{81}$ $2=\sqrt[4]{81}-\frac{81}{81}$ $3=\sqrt[4]{81}$ $4=\sqrt[4]{81}+\frac{81}{81}$ $5=\sqrt{81}-\left(\sqrt[4]{81}+\frac{81}{81}\right)$ $6=\sqrt{81}-\sqrt[4]{81}$

$7=\sqrt{81}-\sqrt[4]{81}+\frac{81}{81}$ $8=\sqrt{81}-\frac{81}{81}$ $9=\sqrt{81}$ $10=\sqrt{81}+\frac{81}{81}$ $11=\sqrt{81}+\sqrt[4]{81}-\frac{81}{81}$

$12=\sqrt{81}+\sqrt[4]{81}$ $13=?$